Reinforcement Learning: Algorithms and Applications

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RL: A Short Introduction
References:

1. Kevin Murphy, “Machine Learning A Probabilistic Perspective”
2. Prof. David Silver’s Course, University College, London, Link: http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html
3. Prof. Sergey Levine’s Course, UC Berkley, Link: http://rail.eecs.berkeley.edu/deeprlcourse-fa18/
4. Prof. Ben Recht’s (UC Berkeley) notes on RL
What is Reinforcement Learning?
What is Machine Learning?
Three Main Classes of ML

1. Unsupervised Learning
2. Supervised Learning
3. Reinforcement Learning
Unsupervised Learning

• **Wikipedia**: “Unsupervised machine learning is the machine learning task of inferring a function that describes the structure of "unlabeled" data (i.e. data that has not been classified or categorized)“

• **Examples**: Clustering, Dimensionality Reduction, Matrix Completion, Image Inpainting, Collaborative Filtering
Unsupervised Learning: Clustering

Figure from [1]
Unsupervised Learning: Clustering

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Unsupervised Learning

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• Examples: Clustering, Dimensionality Reduction, Matrix Completion, Image Inpainting, Collaborative Filtering

• Descriptive analytics refers to summarizing data in a way to make it more interpretable
Supervised Learning

- **Wikipedia**: “Supervised learning is the machine learning task of learning a function that maps an input to an output based on example input-output pairs”
Supervised Learning

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• Examples: Image classification, Handwriting recognition, Email spam filtering, Face recognition, Speech recognition
Supervised Learning: Image Classification

Training Data

airplane
automobile
bird
cat
deer
dog
frog
horse
ship
truck

Testing Data

Cat
Supervised Learning: Face Recognition

Figure from [1]
Supervised Learning

Facebook Outage, July, 2019
Supervised Learning

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- Predictive analytics aims to estimate outcomes from current data
Reinforcement Learning

**Wikipedia:** “Reinforcement learning (RL) is an area of machine learning concerned with how software agents ought to take actions in an environment so as to maximize some notion of cumulative reward. The problem, due to its generality, is studied in many other disciplines, such as game theory, control theory, operations research, information theory, optimization, multi-agent systems, swarm intelligence, statistics, ...”
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RL: Agent and Environment

- At each time step $t$ the **agent:**
  - Executes an action $A_t$
  - Receives reward $R_t$
  - Receives observation $O_{t+1}$

- The **environment:**
  - Receives an action $A_t$
  - Emits reward $R_t$
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- Time $t \leftarrow t+1$
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- Time $t \leftarrow t+1$  

Rules of the game are unknown!
Why RL is Different?
Why RL is Different?

RL: Learning to make a optimal *sequence* of decisions under uncertainty
Why RL is Different?

RL: Learning to make an optimal sequence of decisions under uncertainty

• No supervisor, only a reward signal
• Feedback is delayed, not instantaneous
• Sequential decision making
• Actions have long-term consequences
• Non i.i.d. data
• Agent’s actions affect the subsequent data it receives
Helicopter Maneuvers

Robotic Hand Solving Rubik's Cube

Playing Atari

Robotic Arms

Learning to Walk

Playing Go

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Prescriptive analytics guides actions to take in order to guarantee outcomes.
RL: Exploration vs Exploitation

• Unlike supervised and unsupervised learning, data is not given before
• Agent learns about the environment by trying things out
  • RL in some way a trial-and-error learning
• Agent should learn a good control policy:
  • From its experiences of the environment but without loosing too much reward along the way

• Online decision-making involves a fundamental choice:
  • Exploitation: Make the best decision given current information
  • Exploration: Gather more information to make the best decisions
• The optimal long-term strategy may involve sub-optimal short-term decisions
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**How to balance exploration and exploitation?**
Exploration vs Exploitation: Examples

• Restaurant Selection
  • **Exploitation:** Go to your favorite restaurant
  • **Exploration:** Try a new restaurant

• Online advertisements
  • **Exploitation:** Show the most successful ad
  • **Exploration:** Show a different ad

• Oil Drilling
  • **Exploitation:** Drill at best known location
  • **Exploration:** Drill at new location

• Games
  • **Exploitation:** Play the move you believe is best
  • **Exploration:** Play an experimental move
Reinforcement Learning Problem

• Agent doesn’t know how the environment works
• Agent has to interact with the environment to learn
• Agent gets two feedback:
  • It can observe the state of the environment at each step
  • It gets a reward at each step
• Agent has to learn a control policy
  • Algorithm to select action sequentially
• Agent’s objective is to maximize the expected cumulative reward
Markov Decision Processes
References:

1. Some figures for this lecture are taken from UC Berkeley CS188 course, with permission:
   https://inst.eecs.berkeley.edu/~cs188/fa18/index.html
MDP Example: Grid World

• Agent lives in a grid world
MDP Example: Grid World

- Agent lives in a grid world
- This world is **non-deterministic**
  - Inherent uncertainties in the environment
  - Actions don’t go always go as planned

Figure from [1]
MDP Example: Grid World

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MDP Example: Grid World

• Agent lives in a grid world
• This world is non-deterministic
  • Inherent uncertainties in the environment
  • Actions don’t go always go as planned
• Agent receives rewards each time step
  • Reward will depend on the current state/action

Figure from [1]
Markov Decision Processes (MDP)

- A set of **states**: $x \in X$
- A set of **actions**: $a \in A$
Markov Decision Processes (MDP)

• A set of states: $x \in X$
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• A transition function: $P(x' | x, a)$
  • Probability moving from $x$ to $x'$ if action $a$ is taken
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- **Reward** function: \( R(x, a) \) or \( R(x, a, x') \)
- **A start state** and/or **a terminal state**
- **A very useful model** for approximating real-world systems!

Figure from [1]
Markov Property

• “Markov” generally means that given the present, the future and the past are independent

\[ P(X_{t+1} = x_{t+1} | X_t = x_t, A_t = a_t, \ldots, X_0 = x_0, A_0 = a_0) = \]
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  • A very useful modeling assumption for a large class of real world problems!
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• If you know the current state and current action, history is irrelevant to predict the future states

• Is the world Markov?
  • A very useful modeling assumption for a large class of real world problems!

• Why is it useful?
  • Tremendous reduction in memory/computation
  • History explodes with time. But no need to store the entire history!
MDP Examples

Lunar Lander

Inverted Pendulum

Cart Pole

Videos are from OpenAI Gym
MDP Examples

Robotic Arm 1

Robotic Arm 2

Videos are from OpenAI Gym
Markov Decision Processes

Definition. A Markov Decision Process (MDP) is a tuple $(\mathcal{X}, \mathcal{A}, P, R)$, where,

- $\mathcal{X}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $P$ is a transition probability matrix, $P(x'|x, a) = \mathbb{P}(x_{t+1} = x'|x_t = x, a_t = a)$
- $R$ is a reward function, $R : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$
Control Policy

• Which action to take in each state?
Control Policy

• Which action to take in each state?

• A control policy specifies the action to take given the current state
  • Can be deterministic or stochastic

\[ \pi(a|x) = \mathbb{P}(a_t = a|x_t = x) \]

• Conditional probability of taking action \( a \) given the state \( x \)
Control Policy

• Which action to take in each state?
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  • Can be deterministic or stochastic
    \[ \pi(a|x) = \mathbb{P}(a_t = a | x_t = x) \]
    • Conditional probability of taking action \( a \) given the state \( x \)
• A policy fully defines the behavior of an agent
Value of a Policy
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Figure from [1]
Value of a Policy

• Value of a policy evaluated at state $x$ is the expected cumulative (discounted) rewards obtained by taking action according that policy, starting from $x$

$$V_{\pi}(x) = \mathbb{E} \left[ R(x_0, a_0) + \gamma R(x_1, a_1) + \gamma^2 R(x_2, a_2) + \ldots + \gamma^t R(x_t, a_t) + \ldots \middle| x_0 = x, a_t \sim \pi(\cdot|x_t) \right]$$

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Figure from [1]
1. How to evaluate a policy?

Given a policy $\pi$, how to compute the value function $V_\pi$ of that policy?

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MDP Questions

1. How to evaluate a policy?

   Given a policy \( \pi \), how to compute the value function \( V_\pi \) of that policy?

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   \]

2. How to compute the optimal value function \( V^* \)?

   \[
   V^*(x) = \max_{\pi} V_\pi(x)
   \]
MDP Questions

1. How to evaluate a policy?

Given a policy $\pi$, how to compute the value function $V_\pi$ of that policy?

$$V_\pi(x) = \mathbb{E} \left[ R(x_0, a_0) + \gamma R(x_1, a_1) + \gamma^2 R(x_2, a_2) + \ldots + \gamma^t R(x_t, a_t) + \ldots \middle| x_0 = x, a_t \sim \pi(\cdot|x_t) \right]$$

2. How to compute the optimal value function $V^*$?

$$V^*(x) = \max_\pi V_\pi(x)$$

3. How to compute the optimal policy $\pi^*$?

$$\pi^*(x) = \arg \max_\pi V_\pi(x)$$
Let $V^*$ be the optimal value function. Then $V^*$ satisfies the equation

$$V^*(x) = \max_{a \in \mathcal{A}} \left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \right)$$
Bellman Optimality Equation

Let $V^*$ be the optimal value function. Then $V^*$ satisfies the equation

$$V^*(x) = \max_{a \in A} \left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \right)$$

• Intuition: Suppose $V^*$ is the optimal value function
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Value at state $x$ by taking action $a$ is

$$\left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) V^*(y) \right)$$

But we want to take the best action.
Bellman Operator and Value Iteration

Define the mapping $T : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ as

$$(TV)(x) = \max_{a \in \mathcal{A}} \left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V(y) \right)$$
Bellman Operator and Value Iteration

Define the mapping $T : \mathbb{R}^{nx} \rightarrow \mathbb{R}^{nx}$ as

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This will give a sequence $V_0, V_1, V_2, \ldots$
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- Will the VI converge?
- Will it converge to $V^*$?
- Is $V^*$ unique?
- How fast does it converge?
- How to get $\pi^*$ from $V^*$?
Fixed Point of the Bellman Operator
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Definition 1 (Fixed Point). Let $f : \mathcal{X} \to \mathcal{X}$. $x^*$ is a fixed point of $f$, if $f(x^*) = x^*$
Fixed Point of the Bellman Operator

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Bellman Operator

$$V^*(x) = \max_{a \in \mathcal{A}} \left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \right)$$

Bellman Optimality Equation
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$$V^* = TV^*$$

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**Definition 1** (Fixed Point). Let $f : \mathcal{X} \rightarrow \mathcal{X}$. $x^*$ is a fixed point of $f$, if $f(x^*) = x^*$

$V^*$ is a fixed point of the Bellman Operator $T$

$$V^*(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{Y}} P(y|x, a)V(y))$$

Bellman Optimality Equation

$$V^* = TV^*$$

Bellman Operator

- Computing optimal value function is equivalent to computing the fixed point of the Bellman Operator
Contraction Mapping

**Proposition 2** (Contraction Mapping). \( T \) is a contraction mapping. For any \( V_1, V_2 \in \mathcal{R}^{n_x} \), \( \|TV_1 - TV_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty \)
Contraction Mapping

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Convergence of Value Iteration

\[ \| V_{k+1} - V^* \| \leq \| TV_k - TV^* \| \leq \gamma \| V_k - V^* \| \quad \text{(One step contraction)} \]

\[ \| V_{k+1} - V^* \| \leq \gamma^{k+1} \| V_0 - V^* \| \quad \text{(k+1) step contraction} \]
Contraction, Fixed Point, Convergence
Contraction, Fixed Point, Convergence

\[ V^* = TV^* \]

\[ 45^\circ \]
Contraction, Fixed Point, Convergence

\[ TV \]

\[ V^* = TV^* \]
Contraction, Fixed Point, Convergence

\[ TV \]

\[ T^2V \]

\[ 45^\circ \]

\[ V^* = TV^* \]
Contraction, Fixed Point, Convergence

\[TV \rightarrow T^2V \rightarrow T^3V \rightarrow V^* = TV^*\]
Summary: Computing $V^*$

- Optimal value function $V^*$ satisfies the **Bellman optimality equation**
Summary: Computing $V^*$

- Optimal value function $V^*$ satisfies the Bellman optimality equation

- $V^*$ is the unique fixed point of the Bellman operator $V^* = TV^*$
Summary: Computing $V^*$

• Optimal value function $V^*$ satisfies the Bellman optimality equation

• $V^*$ is the unique fixed point of the Bellman operator $V^* = TV^*$

• Value iteration, $V_{k+1} = TV_k$, converges to $V^*$
Value Iteration
Value Iteration
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Value Iteration

VALUES AFTER 2 ITERATIONS
Value Iteration

VALUES AFTER 3 ITERATIONS
Value Iteration
Value Iteration

VALUES AFTER 5 ITERATIONS

0.51  0.72  0.84  1.00

0.27

0.00  0.22  0.37  0.13

-1.00
Value Iteration

VALUES AFTER 6 ITERATIONS
Value Iteration

VALUES AFTER 7 ITERATIONS

0.62 → 0.74 → 0.85 → 1.00

0.50 ↑ 0.57 ↑ -1.00

0.34 → 0.36 → 0.45 ← 0.24
Value Iteration

VALUES AFTER 8 ITERATIONS

0.63 0.74 0.85 1.00

0.53 ▲

0.57 ▲

-1.00 ▲

0.42 0.39 0.46 ▼

0.26 ▼
Value Iteration

VALUES AFTER 9 ITERATIONS

0.64 0.74 0.85 1.00

0.55          0.57 -1.00

0.46 0.40 0.47 0.27
Value Iteration

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<td>0.74</td>
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VALUES AFTER 10 ITERATIONS
Value Iteration
Value Iteration

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VALUES AFTER 100 ITERATIONS
State-Action Value Function (Q-function)

• Value of a policy:

\[ V_\pi(x) = \mathbb{E}_\pi[R(x_0, \pi(x_0)) + \gamma R(x_1, \pi(x_1)) + \gamma^2 R(x_2, \pi(x_2)) + \gamma^3 R(x_3, \pi(x_3)) + \ldots \mid x_0 = x] \]

• Expected cumulative discounted reward obtained by starting from state \(x\) and following the policy \(\pi\)
State-Action Value Function (Q-function)

• Value of a policy:

\[ V_\pi(x) = \mathbb{E}_\pi[R(x_0, \pi(x_0)) + \gamma R(x_1, \pi(x_1)) + \gamma^2 R(x_2, \pi(x_2)) + \gamma^3 R(x_3, \pi(x_3)) + \ldots \mid x_0 = x] \]

• Expected cumulative discounted reward obtained by starting from state x and following the policy \( \pi \)

• Q-Value of a policy:

\[ Q_\pi(x, a) = \mathbb{E}_\pi[R(x_0, a_0)) + \gamma R(x_1, \pi(x_1)) + \gamma^2 R(x_2, \pi(x_2)) + \gamma^3 R(x_3, \pi(x_3)) + \ldots \mid x_0 = x, a_0 = a] \]

• Expected cumulative discounted reward obtained by starting from state x, taking action a, and then following the policy \( \pi \)
Optimal Q-Function

Q-function for a given policy:

\[ Q_\pi(x, a) = R(x, a) + \gamma \sum_{y \in X} P(y|x, a)V_\pi(y) \]
Optimal Q-Function

Q-function for a given policy:  

\[ Q_\pi(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V_\pi(y) \]

Optimal Q-function:  

\[ Q^*(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \]
Optimal Q-Function

Q-function for a given policy:

\[ Q_\pi(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V_\pi(y) \]

Optimal Q-function:

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How do we get \( V^* \) from \( Q^* \)?
Optimal Q-Function

Q-function for a given policy:  
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Optimal Q-function:  
\[ Q^*(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \]

How do we get \( V^* \) from \( Q^* \)?  
\[ V^*(x) = \max_{a \in \mathcal{A}} Q^*(x, a) \]

Recall optimal Value function:  
\[ V^*(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y)) \]
Optimal Q-Function

Q-function for a given policy:  \[ Q_\pi(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V_\pi(y) \]

Optimal Q-function:  \[ Q^*(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y) \]

How do we get \( V^* \) from \( Q^* \)?

Recall optimal Value function:  \[ V^*(x) = \max_{a \in \mathcal{A}} Q^*(x, a) \]

Recall optimal Value function:  \[ V^*(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V^*(y)) \]

Bellman Equation for \( Q^* \):

\[ Q^*(x, a) = R(x, a) + \gamma \sum_{y} P(y|x, a) \max_{b} Q^*(y, b) \]
Value, Q-value and Policy

$V^* \text{ to } Q^*$:

$Q^*(x, a) = R(x, a) + \gamma \mathbb{E}[V^*(y)|x, a]$ 

$Q^* \text{ to } V^*$:

$V^*(x) = \max_{a \in A} Q^*(x, a)$
Value, Q-value and Policy

Computing optimal policy: \[ \pi^*(x) = \arg \max_a Q^*(x, a) \]

\[ V^* \text{ to } Q^*: \]
\[ Q^*(x, a) = R(x, a) + \gamma \mathbb{E}[V^*(y)|x, a] \]

\[ Q^* \text{ to } V^*: \]
\[ V^*(x) = \max_{a \in A} Q^*(x, a) \]
Policy Iteration

Iteration over policy space to find the optimal policy
Policy Iteration

Iteration over policy space to find the optimal policy

1. Start with an arbitrary policy $\pi_0$
Policy Iteration

Iteration over policy space to find the optimal policy

1. Start with an arbitrary policy $\pi_0$

2. At each iteration $k$
Policy Iteration

Iteration over policy space to find the optimal policy

1. Start with an arbitrary policy $\pi_0$

2. At each iteration $k$
   
   (a) Evaluate policy $\pi_k$ to get $V_{\pi_k}$ (solve the equation $V_{\pi_k} = T_{\pi_k} V_{\pi_k}$)
Policy Iteration

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   (b) Update the policy to get $\pi_{k+1}$

$$
\pi_{k+1}(x) = \arg\max_{a \in A} \left( R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) V_{\pi_k}(y) \right)
$$

Greedy update
Policy Iteration

Iteration over policy space to find the optimal policy

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      $$\pi_{k+1}(x) = \arg \max_{a \in A} (R(x, a) + \gamma \sum_{y \in X} P(y|x, a)V_{\pi_k}(y))$$

      (select $\pi_{k+1}$ s.t. $T_{\pi_{k+1}} V_{\pi_k} = TV_{\pi_k}$)
Policy Iteration

Iteration over policy space to find the optimal policy

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Policy Iteration converges in finite number of steps!
Policy Iteration

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(a) Evaluate policy $\pi_k$ to get $V_{\pi_k}$ (solve the equation $V_{\pi_k} = T_{\pi_k} V_{\pi_k}$)

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$$
\pi_{k+1}(x) = \arg \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{Y}} P(y|x, a)V_{\pi_k}(y))
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Policy Iteration

2. At each iteration $k$
   
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   (b) Update the policy to get $\pi_{k+1}$
   
   $$\pi_{k+1}(x) = \arg\max_{a \in A} (R(x, a) + \gamma \sum_{y \in A'} P(y|x,a)V_{\pi_k}(y))$$
   
   (select $\pi_{k+1}$ s.t. $T_{\pi_{k+1}}V_{\pi_k} = TV_{\pi_k}$)

**Theorem 3.** In Policy Iteration, there exists a finite number $k_0$ such that $V_{\pi_k} = V^*$ for all $k \geq k_0$. 
Policy Iteration

2. At each iteration $k$

   (a) Evaluate policy $\pi_k$ to get $V_{\pi_k}$ (solve the equation $V_{\pi_k} = T_{\pi_k} V_{\pi_k}$)

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   $$\pi_{k+1}(x) = \arg \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{Y}} P(y|x, a)V_{\pi_k}(y))$$

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**Theorem 3.** In Policy Iteration, there exists a finite number $k_0$ such that $V_{\pi_k} = V^*$ for all $k \geq k_0$.

The sequence $\{V_{\pi_0}, V_{\pi_1}, V_{\pi_2}, \ldots\}$ is a monotonically increasing sequence bounded above by $V^*$, and the sequence must therefore converge. As there are finitely many deterministic policies, this convergence must happen in finite time.
Policy Iteration

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**Bad news:** It can take up to $n_{\alpha}^n$ steps
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The sequence $\{V_{\pi_0}, V_{\pi_1}, V_{\pi_2}, \ldots\}$ is a monotonically increasing sequence bounded above by $V^*$, and the sequence must therefore converge. As there are finitely many deterministic policies, this convergence must happen in finite time.

Bad news: It can take up to $n_x^n a^n$ steps 
Good news: Much faster in practice!
Dynamic Programming

1. How to evaluate a policy?
   Given a policy $\pi$, $V_{k+1} = T_{\pi_k} V_k$. Then $V_k \to V_\pi$

2. How to compute the optimal value function $V^*$?
   Value Iteration: $V_{k+1} = TV_k$. Then $V_k \to V^*$

3. How to compute the optimal policy $\pi^*$?
   Value Iteration, Policy Iteration
Q Learning
References:

1. “Reinforcement Learning: An Introduction”, Richard S. Sutton and Andrew G. Barto, Ch. 5, Ch. 6
2. “Algorithms for Reinforcement Learning”, Csaba Szepesvari, Ch. 3
3. “Neuro-Dynamic Programming”, Dimitri P. Bertsekas and John Tsitsiklis, Ch. 5
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3. How to compute the optimal policy $\pi^*$?
   **Value Iteration, Policy Iteration**

Requires the system model $P$
Reinforcement Learning

1. How to evaluate a policy?
2. How to compute the optimal value function $V^*$?
3. How to compute the optimal policy $\pi^*$?

When the system model is unknown
Reinforcement Learning

1. How to evaluate a policy?
2. How to compute the optimal value function $V^*$?
3. How to compute the optimal policy $\pi^*$?

- Need to learn from experiences and observations
  - Observations: Sequences of states, actions, and rewards
Bellman Operator for Q-Function

• Recall Bellman Operator for $V$:

$$(TV)(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V(y))$$

• Bellman Operator for $Q$

$$(FQ)(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) \max_{b \in \mathcal{A}} Q(y, b)$$
Bellman Operator for Q-Function

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\[(TV)(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V(y))\]

• Bellman Operator for Q

\[(FQ)(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) \max_{b \in \mathcal{A}} Q(y, b)\]

**Proposition 1.** (i) $F$ is a monotone mapping. (ii) $F$ is a contraction mapping
Why Q-function?

- Q to action is simple: \( \pi^*(x) = \arg \max_{a \in A} Q^*(x, a) \)
Why Q-function?

• Obtaining unbiased samples of \((FQ)\) is easier than obtaining unbiased samples of \((TV)\)
Why Q-function?

• Obtaining unbiased samples of \((FQ)\) is easier than obtaining unbiased samples of \((TV)\)

\[
(FQ)(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{Y}} P(y|x, a) \max_{b \in \mathcal{A}} Q(y, b)
\]

\[
= R(x, a) + \gamma \mathbb{E} \left[ \max_{b \in \mathcal{A}} Q(x_{t+1}, b) \mid x_t = x, a_t = a \right]
\]
Why Q-function?

- Obtaining unbiased samples of $(FQ)$ is easier than obtaining unbiased samples of $(TV)$

$$(FQ)(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) \max_{b \in \mathcal{A}} Q(y, b)$$

$$= R(x, a) + \gamma \mathbb{E}[\max_{b \in \mathcal{A}} Q(x_{t+1}, b) | x_t = x, a_t = a]$$

$R(x_t, a_t) + \gamma \max_{b \in \mathcal{A}} Q(x_{t+1}, b)$ is an unbiased sample of $(FQ)(x_t, a_t)$ if $x_{t+1} \sim P(\cdot | x_t, a_t)$
Why Q-function?

• Obtaining unbiased samples of (FQ) is easier than obtaining unbiased samples of (TV)

\[(FQ)(x, a) = R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a) \max_{b \in \mathcal{A}} Q(y, b)\]

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\[R(x_t, a_t) + \gamma \max_{b \in \mathcal{A}} Q(x_{t+1}, b) \text{ is an unbiased sample of } (FQ)(x_t, a_t) \text{ if } x_{t+1} \sim P(\cdot|x_t, a_t)\]

\[(TV)(x) = \max_{a \in \mathcal{A}} (R(x, a) + \gamma \sum_{y \in \mathcal{X}} P(y|x, a)V(y))\]

\[= \max_{a \in \mathcal{A}} (R(x, a) + \gamma \mathbb{E}[V(x_{t+1})|x_t = x, a_t = a])\]

\[\max_{a \in \mathcal{A}} (R(x_t, a_t) + \gamma V(x_{t+1})) \text{ is not an unbiased sample of } (TV)(x_t)\]
Q-Learning Algorithm

• Learning the optimal value function directly
  • Action can be taken with respect to a *behavioral* policy
Q-Learning Algorithm

- Learning the optimal value function directly
  - Action can be taken with respect to a behavioral policy

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Q-Learning</th>
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<tbody>
<tr>
<td>1: Initialization: A behavioral policy $\mu$, $t = 0$, initial state $s_t = s_0$</td>
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<tr>
<td>2: <strong>for</strong> each $t = 0, 1, 2, \ldots$ <strong>do</strong></td>
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<td>3: Observe the current state $s_t$</td>
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<td>4: Take action $a_t$ according to the policy $\mu$: $a_t \sim \mu(s_t)$</td>
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<tr>
<td>5: Observe the reward $R_t$ and the next state $s_{t+1}$</td>
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<tr>
<td>6: Update $Q$:</td>
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<tr>
<td>$Q(s_t, a_t) = Q(s_t, a_t) + \alpha (R_t + \gamma \max_b Q(s_{t+1}, b) - Q(s_t, a_t))$</td>
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<tr>
<td>7: <strong>end for</strong></td>
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</table>
Q-Learning Algorithm

Theorem. If: (i) all state-action pairs are visited infinitely often, and (ii) step size satisfies Robbins-Munro condition, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, then Q-Learning will converge, i.e., $Q_t \to Q^*$ almost surely.
Q-Learning Algorithm

Theorem. If: (i) all state-action pairs are visited infinitely often, and (ii) step size satisfies Robbins-Munro condition, \( \sum_t \alpha_t = \infty, \sum_t \alpha_t^2 < \infty \), then Q-Learning will converge, i.e., \( Q_t \to Q^* \) almost surely.

- It is a very surprising result!
  - Convergence to the optimal Q-value even if you are acting sub-optimally (according to a behavioral policy)
Q-Learning Algorithm

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- This is called off-policy learning
Q-Learning Algorithm

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- It is a very surprising result!
  - Convergence to the optimal Q-value even if you are acting sub-optimally (according to a behavioral policy)
- This is called **off-policy learning**
- Caveats:
  - Need to *explore* enough (make sure that behavioral policy will visit all state-action pairs infinitely often)
  - Need to use the correct learning rate
Q-Learning Algorithm: Behavioral Policy

• Q-Learning is an off-policy learning algorithm
  • However, need to explore enough (make sure that behavioral policy will visit all state-action pairs infinitely often)

• Standard behavioral policy: $\epsilon$-greedy policy

\[
\pi_{k+1}(s) = \epsilon\text{-greedy}(Q_k) = \begin{cases} 
  a^* = \arg \max_a Q_k(s, a) & \text{with prob. } 1 - \epsilon \ (|\mathcal{A}| - 1) \\
  a & \text{with prob. } \epsilon
\end{cases}
\]
Q-Learning Algorithm: Behavioral Policy

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• Under some conditions on the transition probability matrix, all state-action pairs will be visited infinitely often (in the limit)
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\quad \text{with prob } \epsilon 
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$$

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- Is $\epsilon$-greedy (Q) is the optimal online control policy?
Q-Learning Algorithm: Behavioral Policy

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  a & \text{with prob. } \epsilon
\end{cases}
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• Under some conditions on the transition probability matrix, all state-action pairs will be visited infinitely often (in the limit)

• Is $\epsilon$-greedy (Q) is the optimal online control policy?

• What do we mean by optimal online control policy?
  Exploration vs Exploitation, Sample Complexity, Safety, Stability
Deep Q Learning
Problems with Large States Space

• RL should be useful to solve large problems:
  • Backgammon: $10^{20}$
  • Go: $10^{170}$ (how many atoms are in the universe?)
  • Helicopter: continuous state
Problems with Large States Space

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  • Go: $10^{170}$ (how many atoms are in the universe?)
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• Recall Q-learning assumptions for convergence

• Will we ever solve such a large scale systems?
Tabular Form to Function Approximation

- We have represented $V$, $Q$, $\pi$ in tabular form
  - One element for each state / (state, action)

- Function Approximation:

\[ Q(s, a) \approx Q(s, a; w) \]

\[ S_t \rightarrow W \rightarrow Q(S_t, A_t; W) \]

Generalize from seen states to unseen states
What Do We Need

• A good class of function approximators
  • Linear combinations of features
  • Decision trees
  • Tile coding
  • Fourier basis
  • Reproducing Kernel Hilbert Spaces
  • Neural networks

• A good training algorithm that is suitable for non-iid and non-stationary data
Q-Learning with FA

- Q-Learning with FA:

\[ w_{t+1} = w_t + \alpha_t \left( R_t + \gamma \max_b Q(s_{t+1}, b, w) - Q(s_t, a_t, w) \right) \nabla Q(s_t, a_t, w) \]
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  • No convergence proofs even with linear function approximation!

"Deadly Triad" according to Sutton and Barto (2018)
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**Function approximation** A powerful, scalable way of generalizing from a state space much larger than the memory and computational resources (e.g., linear function approximation or ANNs).

**Bootstrapping** Update targets that include existing estimates (as in dynamic programming or TD methods) rather than relying exclusively on actual rewards and complete returns (as in MC methods).

**Off-policy training** Training on a distribution of transitions other than that produced by the target policy. Sweeping through the state space and updating all states uniformly, as in dynamic programming, does not respect the target policy and is an example of off-policy training.
Correlation and Forgetting in Q-Learning

• Q-Learning:

1. Take some actions $a_i$ and get the data point $e_i = (s_i, a_i, R_i, s'_i)$

2. Update $w = w + \alpha (R_i + \gamma \max_b Q(s'_i, b, w) - Q(s_i, a_i, w)) \nabla Q(s_i, a_i, w)$
Correlation and Forgetting in Q-Learning

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• Data is used in sequentially, in the same way as they are observed
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  • This breaks the i.i.d. assumption of many popular SGD algorithms

• **Rapid forgetting:** of possibly rare experiences does not effectively use the information (that would be useful later on)
Experience Relay

• Experience Replay:
  • Break the temporal correlations by mixing more and less recent experience for the updates
  • Using rare experience for more than just a single update
Experience Relay

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Necessary to do off-policy learning for ER!

QL-is off-policy learning!
Target Network

• QL gradient update:

\[ w = w + \alpha \left( R_i + \gamma \max_b Q(s'_i, b, w) - Q(s_i, a_i, w) \right) \nabla Q(s_i, a_i, w) \]

Target for gradient update
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• In supervised learning, the target doesn’t depend on the parameter

• In QL, target itself depend on the parameter
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  • This leads to oscillation and instability
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• In QL, target itself depend on the parameter
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• Solution:
  • Use a separate target network
  • Keep the target network unchanged for multiple updates
Target Network

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Target for gradient update

• QL with target network

\[ w = w + \alpha (R_i + \gamma \max_b Q(s'_i, b, w^-) - Q(s_i, a_i, w)) \nabla Q(s_i, a_i, w) \]

\[ w^- = w \text{ after every } N \text{ steps} \]
Deep Q-Networks (DQN) for Atari Games

• **State**: Screen Images (history)
  • 210 x 160 pixel, 128 color

• **Action**: Joystick position
  • 18 different positions

• **Reward**: Game score

• **Objective**: Win the game (a control policy that maximizes game score)

*Mnih et al, Human-level control through deep reinforcement learning, Nature, 2015*
QL with Function Approximation

- Function Approximation

\[ Q(S_t, A_t; W) \]
QL with Function Approximation

- Function Approximation

Use a (deep) NN for function approximation
QL with (Deep) NN

- **Preprocessing**
  - Each original frame is 210 x 160 pixel images with a 128-colour palette
  - Preprocessing: Reduce it to 84 x 84 images
  - History: Use the 4 most recent frame
  - State size = 84 x 84 x 4
QL with (Deep) NN

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• Architecture
  • Input state (s), output Q(s, a), for each \( a \)
    • Avoid forward pass for each possible action
  • Three CNN and two fully connected layers with ReLU
QL with (Deep) NN
QL with (Deep) NN

Same network architecture, hyperparameters for 49 games in Atari
DQN Performance in Atari

• 100 x (DQN score - random play score)/
  (human score - random play score)

Score for each game is averaged over 30 sessions on each game, each lasting up to 5 minutes and beginning with a random initial game state.

The professional human tester played using the same emulator (with the sound turned off). After 2 hours of practice, the human played about 20 episodes of each game for up to 5 minutes each and was not allowed to take any break during this time.

DQN played better than the human player on 22 of the games.

By considering any performance that scored at or above 75% of the human score to be comparable to, or better than, human-level play, Mnih et al. concluded that the levels of play DQN learned reached or exceeded human level on 29 of the 46 games.
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DQN Training

• **Training time**: Training over 50 million frames
  • 38 days of game experience in total
• **Replay memory**: Recent 1 million frames
• **Minibatch size**: 32
• **Target network update frequency**: After every 10k parameter updates
• **Action repeat**: Repeat the same action for \( k (= 4) \) frames
• **SGD**: RMSProp, with learning rate 0.00025
• **Exploration**: Epsilon-greedy policy
  • Epsilon decreasing from 1.0 to 0.1 over first million frames and then fixed after
• **Discount factor**: 0.99
## Effect of Replay and Target Network

<table>
<thead>
<tr>
<th>Game</th>
<th>With replay, with target Q</th>
<th>With replay, without target Q</th>
<th>Without replay, with target Q</th>
<th>Without replay, without target Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Breakout</td>
<td>316.8</td>
<td>240.7</td>
<td>10.2</td>
<td>3.2</td>
</tr>
<tr>
<td>Enduro</td>
<td>1006.3</td>
<td>831.4</td>
<td>141.9</td>
<td>29.1</td>
</tr>
<tr>
<td>River Raid</td>
<td>7446.6</td>
<td>4102.8</td>
<td>2867.7</td>
<td>1453.0</td>
</tr>
<tr>
<td>Seaquest</td>
<td>2894.4</td>
<td>822.6</td>
<td>1003.0</td>
<td>275.8</td>
</tr>
<tr>
<td>Space Invaders</td>
<td>1088.9</td>
<td>826.3</td>
<td>373.2</td>
<td>302.0</td>
</tr>
</tbody>
</table>
## DQN vs Linear

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<th>Linear</th>
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DQN Algorithm

Algorithm 1: deep Q-learning with experience replay.
Initialize replay memory $D$ to capacity $N$
Initialize action-value function $Q$ with random weights $\theta$
Initialize target action-value function $\hat{Q}$ with weights $\theta^- = \theta$

For episode = 1, $M$ do

Initialize sequence $s_1 = \{x_1\}$ and preprocessed sequence $\phi_1 = \phi(s_1)$

For $t = 1, T$ do

With probability $\epsilon$ select a random action $a_t$
otherwise select $a_t = \arg\max_a Q(\phi(s_t), a; \theta)$
Execute action $a_t$ in emulator and observe reward $r_t$ and image $x_{t+1}$
Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $D$
Sample random minibatch of transitions $\left(\phi_j, a_j, r_j, \phi_{j+1}\right)$ from $D$

Set $y_j = \begin{cases} r_j & \text{if episode terminates at step } j+1 \\ r_j + \gamma \max_{a'} \hat{Q}(\phi_{j+1}, a'; \theta^-) & \text{otherwise} \end{cases}$

Perform a gradient descent step on $\left(y_j - Q(\phi_j, a_j; \theta)\right)^2$ with respect to the network parameters $\theta$
Every $C$ steps reset $\hat{Q} = Q$

End For

End For

Code: https://sites.google.com/a/deepmind.com/dqn/
DQN Improvements

• Prioritized Experience Replay (Shaul et al, ICLR 2016)
  • In experience replay, experience transitions were uniformly sampled from a replay memory
  • However, some transitions may be more informative than others, but may occur less frequently
  • Uniform sampling may not select these experiences
  • Can we prioritize the samples to accelerate the learning progress?
  • Intuition: Prioritize a sample based on how much can learn from a transition (expected learning progress)
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• **Double DQN** (Van Hasselt, Guez and Silver, AAAI 2026)
  • Overcome the maximization bias in Q-learning
DQN Improvements

• **Dueling DQN** (Wang et al, ICML 2016)
  • Two streams to separately estimate (scalar) state-value and the advantages for each action

• **Noisy Networks for Exploration** (Fortunato sselt, ICLR 2018)
  • Exploration via adding noise to the neural network parameters

• **Distributional Reinforcement Learning** (Bellemare, ICML, 2018)
  • Tracks the distribution of the Q-values instead of a point estimate

• **Rainbow** (Hessel et al, AAAI 2018)
  • Combining all the DQN improvements
DQN Rainbow Performance

The graph above illustrates the performance of different reinforcement learning algorithms, specifically DQN, DDQN, Prioritized DDQN, Dueling DDQN, A3C, Distributional DQN, Noisy DQN, and Rainbow, across millions of frames. The y-axis represents the median human-normalized score, while the x-axis shows the number of millions of frames. The performance of each algorithm is tracked over time, with Rainbow generally showing the highest scores and the smoothest increases.
Recent Improvements

“Agent57: Outperforming the human Atari benchmark”
https://deepmind.com/blog/article/Agent57-Outperforming-the-human-Atari-benchmark

“We’ve developed Agent57, the first deep reinforcement learning agent to obtain a score that is above the human baseline on all 57 Atari 2600 games”
Policy Gradient Algorithms
Reinforcement Learning

1. How to evaluate a policy?
2. How to compute the optimal value function $V^*$?
3. How to compute the optimal policy $\pi^*$?

- **Q Learning**: Learn optimal value (Q-value) function
  - Compute the policy from the learned Q-function

- Can we learn the policy directly?

When the system model is unknown
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**Policy Gradient Algorithms**

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Parametric Policy

• Consider a (stochastic) policy $\pi_\theta(s, a)$
Parametric Policy

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- Define \( J(\theta) = V_{\pi_\theta} \)
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- **Parametric policy examples:**
  - Softmax policy: \( \pi_\theta(s, a) = \frac{e^{(\phi(s,a)^\top \theta)}}{\sum_b e^{(\phi(s,b)^\top \theta)}} \)
  - Gaussian policy (for continuous action space): \( a \sim \mathcal{N}(\phi(s)^\top \theta, \sigma^2) \)
Policy Gradient

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- Consider a (stochastic) policy $\pi_\theta(s, a)$
- Define $J(\theta) = V_{\pi_\theta}$
- **Goal**: Find $\theta^* = \arg \max_\theta J(\theta)$

- **Policy gradient intuition**: $\theta = \theta + \alpha \nabla J(\theta)$
  - Will this converge?
  - How do we estimate the gradient?
Why Policy Gradient?

• **Advantages:**
  • In many problems, policy may be a simpler function to approximate
  • Choice of policy parameterization is a good way of incorporating domain knowledge about the system into the reinforcement learning algorithm

• **Disadvantages:**
  • Typically converge to a local rather than global optimum
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• How do we estimate the gradient?
Policy Gradient Theorem

**Theorem 2.** For infinite horizon discounted reward MDP,

$$
\nabla J(\theta) = \sum_s \mu_{\pi_\theta}(s) \sum_a Q_{\pi_\theta}(s, a) \nabla \pi_\theta(s, a)
$$

where, $\mu_{\pi_\theta}(s) = \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s | s_0)$. This can be represented as

$$
\nabla J(\theta) = \mathbb{E}_{s \sim \mu_{\pi_\theta}} \mathbb{E}_{a \sim \pi_\theta}(s, \cdot) [Q_{\pi_\theta}(s, a) \nabla \log \pi_\theta(s, a)] = \mathbb{E}_{\pi_\theta} [Q_{\pi_\theta}(s, a) \nabla \log \pi_\theta(s, a)]
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Policy Gradient Theorem

Theorem 2. For infinite horizon discounted reward MDP,

\[ \nabla J(\theta) = \sum_s \mu_{\pi_\theta}(s) \sum_a Q_{\pi_\theta}(s, a) \nabla \pi_\theta(s, a) \]

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\[ \nabla J(\theta) = \mathbb{E}_{s \sim \mu_{\pi_\theta}(\cdot)} \mathbb{E}_{a \sim \pi_\theta(s, \cdot)} [Q_{\pi_\theta}(s, a) \nabla \log \pi_\theta(s, a)] = \mathbb{E}_{\pi_\theta} [Q_{\pi_\theta}(s, a) \nabla \log \pi_\theta(s, a)] \]

• Doesn't depend on the gradient of the (stationary) distribution induced by the policy!
• Can be estimated from the sample trajectories
Actor-Critic Algorithm

• We have \( \nabla J(\theta) = \mathbb{E} [Q_{\pi_{\theta}}(s, a) \nabla \log(\pi_{\theta}(s, a))] \)
Actor-Critic Algorithm

• We have \( \nabla J(\theta) = \mathbb{E} [Q_{\pi_{\theta}}(s, a) \nabla \log(\pi_{\theta}(s, a))] \)

• Approximate Q function: \( Q_w(s, a) \approx Q_{\pi_{\theta}}(s, a) \). Then,

\[
\nabla J(\theta) \approx \mathbb{E} [Q_w(s, a) \nabla \log(\pi_{\theta}(s, a))]
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\[ \theta = \theta + \alpha Q_w(s, a) \nabla \log(\pi_{\theta}(s, a)) \]
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  Yes, but what is the \( w \) to be used?
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• Approximate Q function: $Q_w(s, a) \approx Q_{\pi_\theta}(s, a)$ . Then,

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• Can we do the update

  $\theta = \theta + \alpha Q_w(s, a) \nabla \log(\pi_\theta(s, a))$  Yes, but what is the $w$ to be used?

• $w$ should give a good approximation $Q_w(s, a) \approx Q_{\pi_\theta}(s, a)$
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- We have $\nabla J(\theta) = \mathbb{E} \left[ Q_{\pi_{\theta}}(s, a) \nabla \log(\pi_{\theta}(s, a)) \right]$

- Approximate Q function: $Q_w(s, a) \approx Q_{\pi_{\theta}}(s, a)$. Then,
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- $Q_{\pi_{\theta}}$ is unknown. So, need to learn $w$

Yes, but what is the $w$ to be used?
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  \theta = \theta + \alpha \ Q_w(s, a) \ \nabla \log(\pi_\theta(s, a))
  \]
  Yes, but what is the \( w \) to be used?

- \( w \) should give a good approximation \( Q_w(s, a) \approx Q_{\pi_\theta}(s, a) \)

- \( Q_{\pi_\theta} \) is unknown. So, need to learn \( w \)

- \( \theta \) changing. So, \( w \) should also change
Actor-Critic Algorithm

\[ \nabla J(\theta) \approx \mathbb{E} [Q_w(s, a) \nabla \log(\pi_\theta(s, a))] \]

- **Critic**: Updates Q-value parameter \( w \)
  
  *(Evaluate the policy corresponding to \( \theta \))*

- **Actor**: Update policy parameter \( \theta \)
  
  *(Improve the policy in the direction suggested by critic)*
Actor-Critic Algorithm

\[ \nabla J(\theta) \approx \mathbb{E}[Q_w(s, a) \nabla \log(\pi_\theta(s, a))] \]

- **Critic**: Updates Q-value parameter \( w \) (Evaluate the policy corresponding to \( \theta \))
- **Actor**: Update policy parameter \( \theta \) (Improve the policy in the direction suggested by critic)

Can be done by TD
Actor-Critic Algorithm

\[ \nabla J(\theta) \approx \mathbb{E} \left[ Q_w(s, a) \nabla \log(\pi_\theta(s, a)) \right] \]

- **Critic**: Updates Q-value parameter \( w \) (Evaluate the policy corresponding to \( \theta \))

- **Actor**: Update policy parameter \( \theta \) (Improve the policy in the direction suggested by critic)

```latex
\begin{align*}
\text{for each step do} \\
\theta &= \theta + \alpha_\theta \ Q_w(s, a) \nabla_\theta \log(\pi_\theta(s, a)) \\
\text{Sample the next state } s', \text{ sample the next action } a' &\sim \pi_\theta(s', \cdot) \\
\delta &= R(s, a) + \gamma Q_w(s', a') - Q_w(s, a) \\
w &= w + \alpha_w \delta \nabla_w Q_w(s, a) \\
\text{end for}
\end{align*}
```
Advantage Actor-Critic

• Reduce variance using baseline: \( \nabla J(\theta) = \mathbb{E} [(Q_{\pi_\theta}(s, a) - b(s)) \nabla \log(\pi_\theta(s, a))] \)
Advantage Actor-Critic

• Reduce variance using baseline: \( \nabla J(\theta) = \mathbb{E} \left[ (Q_{\pi_\theta}(s, a) - b(s)) \nabla \log(\pi_\theta(s, a)) \right] \)

• Value function is a good baseline: \( \nabla J(\theta) = \mathbb{E} \left[ (Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s)) \nabla \log(\pi_\theta(s, a)) \right] \)
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Advantage Function
Advantage Actor-Critic

• Reduce variance using baseline: \( \nabla J(\theta) = \mathbb{E} \left[ (Q_{\pi_\theta}(s, a) - b(s)) \nabla \log(\pi_\theta(s, a)) \right] \)

• Value function is a good baseline: \( \nabla J(\theta) = \mathbb{E} \left[ (Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s)) \nabla \log(\pi_\theta(s, a)) \right] \)

\[ \nabla J(\theta) = \mathbb{E} \left[ A_{\pi_\theta}(s, a) \nabla \log(\pi_\theta(s, a)) \right] \]
Advanced Policy Gradient Algorithms

• Natural Policy Gradient Algorithm (Kakade, NIPS, 2002)

• Trust Region Policy Optimization (TRPO) (Schulman et al, ICML, 2015)

• Proximal Policy Optimization (PPO) (Schulman et al, 2017)
Some Examples with PPO

Trained by Rayan El Helo (ECE, TAMU)
Some Examples with PPO

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Taxonomy of RL Algorithms

Conclusion

- RL provides a general-purpose framework for AI
- RL problems can be solved by end-to-end deep learning
- Reinforcement Learning + Deep Learning = AI?
Conclusion

• **RL is a very active research area!!**
  - How do we learn fast? (RL is infamous for being data hungry)
  - How do we learn safely? (Don’t want my drone to crash during training/testing)
  - How do we use memory for transfer/meta learning? (Learning one task should be useful to execute other tasks)
  - How do we represent and learn hierarchical features? (Breaking down a very large task to simple tasks, to reduce the complexity)
  - How do we learn from an expert? (Can the AI agent learn from human agents’ demonstrations?)
  - How do we learn with multiple agents? (Especially if the agents are rational and selfish)

• Would you like to do some cool research on RL?
  • Please send me an email!